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New seventh and eighth order derivative free methods for solving nonlinear equations

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Abstract

The purpose of this work is to develop two new iterative methods for solving nonlinear equations which does not require any derivative evaluations. These composed formulae have seventh and eighth order convergence and desire only four function evaluations per iteration which support the Kung-Traub conjecture on optimal order for without memory schemes. Finally, numerical comparison is provided to show its effectiveness and performances over other similar iterative algorithms in high precision computation.

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1 Introduction

The problem of finding the real roots of the nonlinear equation

$$f(x) = 0, \tag{1.1}$$

is the most fascinating computational problems in numerical analysis. The solution of nonlinear equations is required for many practical situations occurs in Physics, Chemistry, Mathematics and Engineering. In most of the cases, the exact solution of nonlinear equations are rarely obtainable. In such cases, we can produce the approximate solution by iterative methods. Iterative methods can be divided into single-point and multi-point schemes. These schemes are further sub-classified into with and without derivative category. The multi-point methods have gain more importance over single point methods because they have high order of convergence and efficiency index. Some times $f(x)$ does not have a derivative or computation of the derivative of $f(x)$ is very cumbersome. Kung and Traub [1] conjectured that an optimal iterative methods consuming $n+1$ function evaluations per iteration could achieve convergence order of 2^n. So the methods which does not use derivative of $f(x)$ is generally more desirable as it reduces the number of function evaluations per iteration. A few of the derivative-free algorithms were discussed in the research articles [2]- [7].

Choosing a good initial approximation of iterative methods for solving nonlinear equations is also interesting problem. If, it is ensure that the initial approximation is close enough to the solution, which gives guarantee to converge the solution. There are many strategies used for finding good initial approximation. In the beginning, the well-known bisection method is mostly used to find the good initial approximation but it is not sufficiently efficient technique. So, after that Yun [8], present a non-iterative method based on numerical integration method briefly referred as NIM, which involved signm, tanh and arctan functions. These three functions are used to find a

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good initial approximation for nonlinear equations. The considered three kinds of so-called sigmoid transformations of f are S-transformations, T-transformation and A- transformation. These NIM does not required any derivative and no iterative process. Using numerical integration, we can obtain relatively good approximation to the sought zero in one step. Furthermore, we present some basic definitions and their references such as [9]-[12].

Definition 1: Let $\alpha, x_n \in R$, $n = 0, 1, 2, ...$ then the sequence $\{x_n\}$ is said to converge to α if

$$lim_{n\to\infty}|x_n - \alpha| = 0.$$

If in addition, there exist a constant $C > 0$, an integer $n_0 > 0$, and $p \geqslant 0$, such that for all $n > n_0$

$$|x_{n+1} - \alpha| \leq C|x_n - \alpha|^p,$$

then $\{x_n\}$ is said to be converges to the root α with order at least p. If $p = 2$ or 3, the convergence is said to be quadratic or cubic, respectively. Here e_n is the error at the n^{th} iteration and the relation

$$e_{n+1} \leq Ce_n^p + O(e_n^{p+1}), \tag{1.2}$$

is called the error equation. The value of p is called the order of convergence.

Definition 2: The efficiency index of an iterative method of order p requiring n function evaluations per iteration is most frequently calculated in Traub sense [12], which is defined by

$$E = p^{1/n}. \tag{1.3}$$

Definition 3: Suppose that x_{n-1}, x_n and x_{n+1} are three successive iterations closer to the root α. Then computational order of convergence (COC) of methods are approximated by [9]

$$COC \approx \frac{ln|f(x_{n+1})/f(x_n)|}{ln|f(x_n)/f(x_{n-1})|}, \tag{1.4}$$

In this paper our main goal is to developed two derivative free without memory methods of order seven and eight, respectively using four function evaluations at each iteration, which is an extended version of the derived method by Mirzaee and Hamzeh[13]. The efficiency index of seventh and eighth order derivative free method are 1.565 and 1.6817, respectively. The article is organized as follows. In section 2, we mention the sub-steps and gives theoretical result for new seventh order derivative free three-step without memory scheme. In section 3, we shows the analytical proof for optimal eighth order derivative free scheme. Moreover, In Section 4, we presented a thorough numerical comparison between the existing derivative free methods and new proposed methods. In section 5, some concluding remarks are given.

2 Seventh Order Iterative Method

We acknowledge the single variate version of the scheme presented by Mirzaee and Hamzeh [13]

$$\begin{aligned}
y_n &= x_n - \frac{f(x_n)}{f'(x_n)},\\
z_n &= x_n - \frac{f(x_n)}{f'(x_n)}\frac{f(y_n) - f(x_n)}{(2f(y_n) - f(x_n))},\\
x_{n+1} &= z_n - \frac{f(z_n)f(x_n)(2f(y_n) - f(x_n))}{f'(x_n)[4f(y_n)f(x_n) - 2f(y_n)^2 - f(x_n)^2]}.
\end{aligned} \tag{2.1}$$

It's convergence rate is six, which includes three function and one derivative evaluations per iteration. So, we can increase the order of convergence of this scheme upto optimal eighth order on the basis of Kung and Traub conjecture. Using the equation (1.3), the efficiency index of above scheme is 1.565. In [14], we accelerated the rate of convergence from six to seven and eight. Sometimes derivative evaluations are more complicated during computation, so to overcome this complication, we present derivative free version of the method. During this approach, we eliminate the derivative evaluation of functions as follows **Method 1**:

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f[x_n, w_n]}, \\ z_n &= x_n - \frac{f(x_n)}{f[x_n, w_n]} A(t_n), \\ x_{n+1} &= z_n - \frac{f(z_n)}{f[x_n, w_n]} \{B(t_n) * H(u_n)\}, n = 0, 1, 2, ... \end{aligned} \tag{2.2}$$

where $A(t_n)$, $B(t_n)$ and $H(u_n)$ are weight function and $t_n = \frac{f(y_n)}{f(x_n)}$, $u_n = \frac{f(z_n)}{f(y_n)}$. Here we use forward difference to approximate the value of the derivative $f'(x_n) \approx f[x_n, w_n] = \frac{f(w_n) - f(x_n)}{(w_n - x_n)}$, where $f(w_n) = x_n + f(x_n)^2$. The first result statement is as follows

Theorem 2.1. Assume $\alpha \in D$ be simple zero of a sufficiently differentiable function $f : D \subseteq R \to R$ for an open interval D which contains x_0 as an initial approximation of α. Then the order of convergence of method defined in (2.2) is seven if
$A(0) = 1,\ A'(0) = 1,\ A''(0) = 4,\ A^{(3)}(0) = 30,\ B(0) = 1,\ H(0) = 1,\ B'(0) = 2,\ H'(0) = 1,$ $B''(0) = 12$
and the error equation is given by

$$\begin{aligned} e_{n+1} &= \frac{1}{24} c_2^2((f'(\alpha))^2 c_2 + c_3)(24(f'(\alpha))^2 c_2 + 48c_3 + c_2^2(-144 + 4B^{(3)}(0) \\ &\quad -A^{(4)}(0)))e_n^7 + O(e_n^8). \end{aligned} \tag{2.3}$$

Proof. Suppose α be the simple zero of $f(x)$ and $f'(\alpha) \neq 0$. We introduced the error equation at n^{th} iteration as $e_n = x_n - \alpha$. Using Taylor series in each term involved in (2.2) about the simple zero α , we get

$$f(x_n) = f'(\alpha)[e_n + c_2 e_n^2 + c_3 e_n^3 + c_4 e_n^4 + c_5 e_n^5 + ... + O(e_n^8)], \tag{2.4}$$

where $c_i = \frac{f^{(i)}(\alpha)}{i! f'(\alpha)}$, $i = 2, 3...$, Moreover, we obtained

$$\begin{aligned} w_n &= x_n + f(x_n)^2 \\ &= e_n + f'(\alpha)^2 [e_n + e_n^2 c_2 + e_n^3 c_4 + e_n^4 c_4 + e_n^5 c_5 + ... + O(e_n^8)]^2, \end{aligned} \tag{2.5}$$

And then

$$\begin{aligned} f(w_n) &= (e_n + f'(\alpha)^2 (e_n + e^2 c_2 + e_n^4 c_4 + e_n^5 c_5 + ...)^2 \\ &\quad + c_2(e_n + f'(\alpha)^2 (e_n + e^2 c_2 + e_n^4 c_4 + e_n^5 c_5 + ...)^2)^2 + \\ &\quad ... + O(e_n^8)). \end{aligned}$$

So, we find the value

$$\begin{aligned} f[x_n, w_n] &= \frac{f(w_n)-f(x_n)}{(w_n-x_n)} = (-f'(\alpha)(e_n + e_n^2 c_2 + e_n^3 c_3 + ...) \\ &+f'(\alpha)(e_n + f'(\alpha)^2(e_n + e_n^2 c_2 + e_n^3 c_3 + ...)^2 \\ &+c_2(e_n + f'(\alpha)^2((e_n + e_n^2 c_2 + e_n^3 c_3 + ...)^2)^2 ... + O(e_n^8). \end{aligned} \tag{2.6}$$

From (2.4) and (2.6), we have

$$\begin{aligned} y_n &= c_2 e_n^2 + (f'(\alpha)^2 c_2 - 2c_2^2 + 2c_3)e_n^3 \\ &+(-f'(\alpha)^2 c_2^2 + 4c_2^3 + 4c_2^3 + 3f'(\alpha)^2 c_3 - 7c_2 c_3 + 3c_4)e_n^4 \\ &+(-f'(\alpha)^4 c_2^2 + 3f'(\alpha)^2 c_2^3 - 8c_2^4 + f'(\alpha)^4 c_3 - 6f'(\alpha)^2 c_2 c_3 \\ &+20c_2^2 c_3 - 6c_3^2 + 6f'(\alpha)^2 c_4 - 10c_2 c_4 + c_5)e_n^5 + ... + O(e_n^8). \end{aligned}$$

As well as

$$\begin{aligned} & f(y_n) = [f'(\alpha)c_2 e_n^2 + f'(\alpha)(f'(\alpha)^2 c_2 - 2c_2^2 + 2c_3)e_n^3 \\ & +f'(\alpha)(-f'(\alpha)^2 c_2^2 + 5c_2^3 + 3f'(\alpha)^2 c_3 - 7c_2 c_3 + 3c_4)e_n^4 \\ & +f'(\alpha)(-f'(\alpha)^4 c_2^2 + 3f'(\alpha)^2 c_2^3 - 8c_2^4 + f'(\alpha)^4 c_3 \\ & -6f'(\alpha)^2 c_2 c_3 + 20c_2^2 c_3 - 6c_3^2 + 2c_2^2(f'(\alpha)^2 c_2 - 2c_2^2 + 2c_3) \\ & +6f'(\alpha)^2 c_4 - 10c_2 c_4 + 4c_5)e_n^5 + ... + O(e_n^8)]. \end{aligned} \tag{2.7}$$

Then, from (2.4) and (2.7), we get

$$\begin{aligned} & \frac{f(y_n)}{f(x_n)} = f'(\alpha)c_2 e_n^2 + f'(\alpha)(f'(\alpha)^2 c_2 - 2c_2^2 + 2c_3)e_n^3 \\ & +f'(\alpha)(-f'(\alpha)^2 c_2^2 + 5c_2^3 + 3f'(\alpha)^2 c_3 - 7c_2 c_3 + 3c_4)e_n^4 + f'(\alpha) \\ & (-f'(\alpha)^4 c_2^2 + 3f'(\alpha)^2 c_2^3 - 8c_2^4 + f'(\alpha)^4 c_3 - 6f'(\alpha)^2 c_2 c_3 \\ & +20c_2^2 c_3 - 6c_3^2 + 2c_2^2(f'(\alpha)^2 c_2 - 2c_2^2 + 2c_3) \\ & +6f'(\alpha)^2 c_4 - 10c_2 c_4 + 4c_5)e_n^5 ... + O(e_n^8). \end{aligned} \tag{2.8}$$

Substituting the values of (2.8) in the second sub-step of method (2.2), we obtain

$$\begin{aligned} z_n - \alpha &= (1 - A(0))e_n + c_2(A(0) - A'(0))e_n^2 \\ &+(f'(\alpha)^2 c_2(A(0) - A'(0)) + 2c_3(A(0) - A'(0)) \\ &-\frac{1}{2}c_2^2(4A(0) - 8A'(0) + A"(0)))e_n^3 + ... + O(e_n^8). \end{aligned}$$

Now, by imposing these conditions $A(0) = 1$, $A'(0) = 1$ and $A''(0) = 4$ in the above equation, we get possible order of convergence is four and by using it, we can get

$$\begin{aligned} f(z_n) &= -\frac{1}{6}\{f'(\alpha)c_2(6f'(\alpha)^2c_2+6c_3+c_2^2(-30+A^{(3)}(0)))\}e_n^4 \\ &+\{f'(\alpha)(-c_2^3-2c_2(3f'(\alpha)^2c_3+c_4)-c_2^2(f'(\alpha)^4+c_3(-32 \\ &+A^{(3)}(0)))-\frac{1}{2}f'(\alpha)^2c_2^3(-28+A^{(3)}(0))+c_2^4(-36+\frac{5}{3}A^{(3)}(0) \\ &-\frac{1}{24}A^{(4)}(0)\}e^5+\{f'(\alpha)(-c_3(9f'(\alpha)^2c_3+7c_4)-c_2(7f'(\alpha)^4c_3 \\ &+11f'(\alpha)^2c_4+3c_5+2c_3^2(-33+A^{(3)}(0)))+\frac{1}{2}c_2^2(3c_4(-32 \\ &+A^{(3)}(0))+7f'(\alpha)^2c_3(-28+A^{(3)}(0)))+\frac{1}{6}f'(\alpha)^2c_2^3(66 \\ &-3A^{(3)}(0)+2c_3(-786+37A^{(3)}(0)-A^{(4)}(0)))+\frac{1}{6}f'(\alpha)c_2^4 \\ &(-504+28A^{(3)}(0)-A^{(4)}(0))-\frac{1}{120}c_2^5(-20400+1240A^{(3)}(0) \\ &-65A^{(4)}(0)+A^{(5)}(0)))\}e_n^6+...+O(e_n^8). \end{aligned} \tag{2.9}$$

Considering (2.7) and (2.9), we have

$$\begin{aligned} \frac{f(z_n)}{f(y_n)} &= \{\frac{1}{6}(-6c_3-c_2(6f'(\alpha)^2+c_2(-30+A^{(3)}(0))))\}e_n^2 \\ &+\{-3f'(\alpha)^2c_3-2c_4+c_2^2(7-\frac{1}{3}f'(\alpha)^2A^{(3)}(0))-\frac{2}{3}c_2c_3 \\ &(-30+A^{(3)}(0))+c_2^3(-26+\frac{4}{3}A^{(3)}(0)-\frac{1}{24}A^{(4)}(0))\}e_n^3 \\ &+\{-f'(\alpha)^4c_3-3(2f'(\alpha)^2c_4+c_5)-\frac{1}{3}c_2(3c_4(-29+A^{(3)}(0)) \\ &+5f'(\alpha)^2c^3(-21+A^{(3)}(0)))+c_3^2(19-\frac{2}{3}A^{(3)}(0)) \\ &+\frac{1}{8}f'(\alpha)^2c_2^3(-272+20A^{(3)}(0)-A^{(4)}(0))+\frac{1}{12}c_2^2(2f'(\alpha)^4 \\ &(-18+A^{(3)}(0))+c_3(-86A^{(3)}(0)+3(520+A^{(4)}(0)))) \\ &-\frac{1}{120}c_2^4(-111160+820A^{(3)}(0)-55A^{(4)}(0)+A^{(5)}(0))\}e_n^4 \\ &+...+O(e_n^8). \end{aligned} \tag{2.10}$$

Now using Taylor series for the simple root in the last sub-step of the method (2.2) and substitute the values of functions from equations (2.6), (2.9) and (2.10) in the scheme (2.2), we get

$$
\begin{aligned}
&(z_n-\alpha)-\frac{f(z_n)}{f[x,w]}B(t_n)H(u_n)\\
&=\{\frac{1}{6}(-1+B(0)H(0))c_2(6f'(\alpha)^2c_2+6c_3+c_2^2(-30+A^{(3)}(0))\}e_n^4\\
&+\{(2(-1+B(0)H(0))c_3^2+2(-1+B(0)H(0))c_2(3f'(\alpha)^2c_3+c_4)\\
&+c_2^2(f'(\alpha)(-1+B(0)H(0)+c_3(32+H(0)(2+B(0)(-34+A^{(3)}(0)))\\
&-A^{(3)}(0)))+\frac{1}{2}f'(\alpha)^2c_2^3(28+H(0)(4+B(0)(-32+A^{(3)}(0)))-A^{(3)}(0))\\
&-\frac{1}{24}c_2^4(864-40A^{(3)}(0)+H(0)(-8(-30+A^{(3)}(0))+B(0)(-1104\\
&+48A^{(3)}(0)-A^{(4)}(0)))+A^{(4)}(0)))e_n^5\\
&\frac{1}{360}\{360(-1+B(0)H(0))c_3(9f'(\alpha)^2c_3+7c_4)+360c_2(7f'(\alpha)^4(-1\\
&+B(0)H(0))(11f'(\alpha)^2c_4+3c_5)+c_3^2(66+8H(0)-2A^{(3)}(0)+B(0)\\
&(-1+(-73+2A^{(0)}(0)H(0)))))+180c_2^2(c_4(96-3A^{(3)}(0)+H(0)(8\\
&+B(0)(-104+3A^{(3)}(0)))f'(\alpha)^2c_3+c_3(196+36H(0)-7A^{(3)}(0)\\
&+B(0)(-4+H(0)(-228+7A^{(3)}(0)))))+60f'(\alpha)^2c_2^4(504+H(0)\\
&(252-8A^{(3)}(0))-28A^{(3)}(0)+B(0)(2(-30+A^{(3)}(0))+H(0)(-726\\
&+35A^{(3)}(0)-A^{(4)}(0)))+A^{(4)}(0))-60c_2^3(3f'(\alpha)^4(-22-8H(0)\\
&+B(0)(2-H(0)(-28+A^{(3)}(0)))+A^{(3)}(0))+c_3(B(0)(2(-30\\
&+A^{(3)}(0))+H(0)(-2070+89A^{(3)}(0)-2A^{(4)}(0)))+2(786\\
&-8H(0)(-33+A^{(3)}(0))-37A^{(3)}(0)+A^{(4)}(0))))+c_2^5(-3(-20400\\
&+1240A^{(3)}(0)+10H(0)(-1104+48A^{(3)}(0)-A^{(4)}(0))-65A^{(4)}(0)\\
&+A^{(5)}(0))+B(0)(-10(-30+A^{(3)}(0))^2+3H(0)(-31440\\
&+1720A^{(3)}(0)-75A^{(4)}(0)+A^{(5)}(0))))\}e_n^6+...+O(e_n^8).
\end{aligned}
\tag{2.11}
$$

For achieving the maximum possible convergence rate of the method (2.2), we put the values of $A^{(3)}(0)=30$, $B(0)=1$, $H(0)=1$, $B'(0)=2$, $H'(0)=1$, $B''(0)=12$ in the above equation and obtain the error equation for the method (2.2) as

$$
\begin{aligned}
e_{n+1} \;=\; &\frac{1}{24}c_2^2(f'(\alpha)^2c_2+c_3)(24f'(\alpha)^2c_2+48c_3+c_2^2(-144+4B^{(3)}(0)\\
&-A^{(4)}(0)))e_n^7)+O(e_n^8).
\end{aligned}
$$

This proves the result. Q.E.D.

3 Optimal order of convergence

In this ensuing section, we will improve the order of convergence from seven to optimal order eight. To serve this purpose, we consider $f(w_n)=x_n+f(x_n)^3$ for obtaining the optimal order of convergence. Here if we increase the degree of $f(x_n)$ more than three, it was observed that the

order will not increase. Now, we formulate the algorithms as follows
Method 2:

$$y_n = x_n - \frac{f(x_n)}{f[x_n, w_n]},$$
$$z_n = x_n - \frac{f(x_n)}{f[x_n, w_n]} A(t_n),$$
$$x_{n+1} = z_n - \frac{f(z_n)}{f[x_n, w_n]} \{B(t_n) * H(u_n) * G(s_n)\}, n = 0, 1, 2, ... \quad (3.1)$$

where $G(s_n)$ is a weight function and $s_n = \frac{f(z_n)}{f(x_n)}$. Here we give the proof of the following theorem:

Theorem 3.1. Suppose that the function $f : D \subseteq R \to R$ for an open interval D has a simple root $\alpha \in D$. Let $f(x)$ be sufficiently smooth in the interval D and the initial guess x_0 is sufficiently close to α. Then the order of convergence of the derivative free iterative scheme defined by (3.1) is eight if $A(0) = 1$, $A'(0) = 1$, $A''(0) = 4$, $A^{(3)}(0) = 30$, $A^{(4)}(0) = 0$, $B(0) = 1$, $H(0) = 1$, $B'(0) = 2$, $H'(0) = 1$, $B''(0) = 12$, $B^{(3)}(0) = 36$, $G(0) = 1$, $G'(0) = 2$, and its error equation is give by

$$\begin{aligned} e_{n+1} &= \frac{1}{120} c_2 c_3 (-120 c_2^2 (f'(\alpha)^3 - 4c_3) - 120 c_2 c_4 + 60 c_3^2 (-2 + H^{(2)}(0)) \\ &\quad + c_2^4 (5B^{(4)}(0) - A^{(5)}(0))) e_n^8 + O(e_n^9). \end{aligned} \quad (3.2)$$

Proof. The proof of this theorem is similar to the previous one. So, we only find the required expressions, the first one is

$$\begin{aligned} \frac{f(z_n)}{f(x_n)} &= \{-c_2 c_3\} e_n^3 + \{\frac{1}{24}(-24 f'(\alpha)^3 c_2^2 + 336 c_2^4 + 72 c_2^2 c_3 \\ &-48 c_3^2 - 48 c_2 c_4 - c_2^4 A^{(4)}(0)\} e_n^4 + \frac{1}{120}\{-120 f'(\alpha)^3 \\ &c_3^2 - 18480 c_2^5 - 720 f'(\alpha)^3 c_2 c_3 + 12600 c_2^3 c_3 + 1080 c_2 c_3^2 \\ &+600 c_2^2 c_4 - 840 c_3 c_4 - 360 c_2 c_5 + 70 c_2^5 A^{(4)}(0) \\ &-40 c_2^3 c_3 A^{(4)}(0) - c_2^5 A^{(5)}(0))\} e_n^5 \end{aligned}$$

$$\begin{aligned} &+\frac{1}{720}\{-720 f'(\alpha)^6 c_2^2 + 40320 f'(\alpha)^3 c_2^4 + 730080 c_2^6 \\ &-5040 f'(\alpha)^3 c_2^2 c_3 - 1007280 c_2^4 c_3 - 6480 f'(\alpha)^3 c_3^2 \\ &+218880 c_2^2 c_3^2 + 4320 c_3^3 - 7920 f'(\alpha)^3 c_2 c_4 + 114480 c_2^3 c_4 \\ &+18720 c_2 c_3 c_4 - 4320 c_4^2 + 5040 c_2^2 c_5 - 7200 c_3 c_5 \\ &-2880 c_2 c_6 - 120 c_2^4 A^{(4)}(0) - 3420 c_2^6 A^{(4)}(0) \\ &+3930 c_2^4 c_3 A^{(4)}(0) - 720 c_2^2 c_3^2 A^{(4)}(0) - 360 c_2^3 c_4 A^{(4)}(0) \\ &+102 c_2^6 A^{(5)}(0) - 60 c_2^4 c_3 A^{(5)}(0) - c_2^6 A^{(6)}(0)\} e_n^6 \\ &+ ... + O(e_n^9), \end{aligned} \quad (3.3)$$

Together with (2.9), (3.3), in the last sub-step of iterative scheme (3.1), we obtain the error equation

$$\begin{aligned} e_{n+1} &= (z_n-\alpha)-\frac{f(z_n)}{f[x_n,z_n]}(B(t_n)*H(u_n)*G(s_n)) \\ &= \{(-1+G(0))c_2c_3\}e_n^4 \\ &+\frac{1}{24}\{(-1+G(0))(24c_2^2(f'(\alpha)^3-2c_3)+48c_3^2+48c_2c_4 \\ &+c_2^4(-336+A^{(4)}(0))\}e_n^5 \\ &+\frac{1}{120}\{(-1+G(0))(360c_2^2c_4+840c_3c_4+360c_2(2f'(\alpha)^3c_3 \\ &+2c_3^2-c_5)-40c_2^3(6f'(\alpha)^3+c_3(-324+A^{(4)}(0)))+c_2^5 \\ &(-16800-65A^{(4)}(0)-A^{(5)}(0))\}e_n^6 \\ &+\{-(-1+G(0))(-9f'(\alpha)^3c_3^2+4c_3^3-6c_4^2-10c_3c_5) \\ &-(-1+G(0))c_2((-11f'(\alpha)^3+16c_3)c_4-4c_6)+\frac{1}{2}(-1 \\ &+G(0))c_2^3c_4(-328+A^{(4)}(0))+c_2^2(14f'(\alpha)^3(-1+G(0))c_3 \\ &+(-1+G(0))(-f'(\alpha)^6+4c_5)-c_3^2(-316+G'(0)-G(0)(-314 \\ &+A^{(4)}(0))+\frac{1}{24}c_2^4(4f'(\alpha)^3(-1+G(0))(-330+A^{(4)}(0)) \\ &c_3(-30720+122A^{(4)}(0)-2A^{(5)}(0)+G(0)(30576+4B^{(3)}(0) \\ &-123A^{(4)}(0)+2A^{(5)}(0))))+\frac{1}{720}(-1+G(0))c_2^6(-619200 \\ &+3000A^{(4)}(0)-96A^{(5)}(0)+A^{(6)}(0))\}e_n^7 \\ &+...+O(e_n^9). \end{aligned} \tag{3.4}$$

This is clearly shows that the weight function in (3.1) must be chosen as stated in the theorem to make it optimal and have the following error equation

$$\begin{aligned} e_{n+1} &= \frac{1}{120}c_2c_3(-120c_2^2(f'(\alpha)^3-4c_3)-120c_2c_4+60c_3^2(-2+H^{(2)}(0)) \\ &+c_2^4(5B^{(4)}(0)-A^{(5)}(0)))e_n^8+O(e_n^9). \end{aligned} \tag{3.5}$$

Q.E.D.

Remark: By using the value of $w_n = x_n + \beta f(x_n)^m$, where $\beta \in R - \{0\}$ and $m \in N$. If we choose the value of $m = 2$, the error expression of the method (2.2) is obtained as

$$\begin{aligned} e_{n+1} &= \frac{1}{24}c_2^2((f'(\alpha))^2c_2\beta+c_3)(24(f'(\alpha))^2c_2\beta+48c_3+c_2^2(-144+4B^{(3)}(0) \\ &-A^{(4)}(0)))e_n^7+O(e_n^8). \end{aligned} \tag{3.6}$$

Furthermore, we choose the of value $m = 3$ and find the error equation of the method (3.1) as

$$\begin{aligned} e_{n+1} &= \frac{1}{120}c_2c_3(-120c_2^2(f'(\alpha)^3\beta-4c_3)-120c_2c_4+60c_3^2(-2+H^{(2)}(0)) \\ &+c_2^4(5B^{(4)}(0)-A^{(5)}(0)))e_n^8+O(e_n^9). \end{aligned} \tag{3.7}$$

So, the varying parameter β does not affect the order of convergence of the scheme.

4 Numerical comparison

In order to demonstrate the accuracy of a method, it is necessary to compare the numerical results of the presented method along with the methods available in the literature. The performances of the proposed methods are compared with some of the existing algorithms. Here we denote the seventh order derivative free method(eq:12) given in [3] by SKSM and eighth order derivative free method (eq:19) presented in [3] by SKEM. Thereafter, we also compare our eighth order derivative free method with the similar nature scheme given by Thukral in [2], which is denoted by TEM. The sub-steps of of these schemes are follows

Soleymani and Khattri seventh order derivative free method (SKSM):

$$
\begin{aligned}
y_n &= x_n - \frac{f(x_n)}{f[x_n, w_n]},\\
z_n &= y_n - \frac{f(y_n)}{f[x_n, w_n]}\left(1 + \frac{f(y_n)}{f(x_n)} + \frac{f(y_n)}{f(w_n)}\right),
\end{aligned}
$$

$$
\begin{aligned}
x_{n+1} &= z_n - \frac{f(z_n)}{f[x_n, w_n]}\\
&\left[1 + (2 - f[x_n, w_n])\frac{f(y_n)}{f(w_n)} + \left[\frac{1}{1 - f[x_n, w_n]}\right]\right.\\
&\left.\left(\frac{f(y_n)}{f(x_n)}\right)^2 + \frac{f(z_n)}{f(y_n)}\right],
\end{aligned}
\tag{4.1}
$$

Soleymani and Khattri eight order derivative free method (SKEM):

$$
\begin{aligned}
y_n &= x_n - \frac{f(x_n)}{f[x_n, w_n]},\\
z_n &= y_n - \frac{f(y_n)}{f[x_n, w_n]}\left(1 + \frac{f(y_n)}{f(x_n)} + \frac{f(y_n)}{f(w_n)}\right),\\
x_{n+1} &= z_n - \frac{f(z_n)}{f[x_n, w_n]}[A_1],
\end{aligned}
\tag{4.2}
$$

where

$$
\begin{aligned}
A_1 \quad &= 1 + (2 - f[x_n, w_n])\frac{f(y_n)}{f(w_n)} + (1 - f[x_n, w_n])\left(\frac{f(y_n)}{f(w_n)}\right)^2\\
&+(-4 + f[x_n, w_n](6 + f[x_n, w_n](-4 + f[x_n, w_n])))\left(\frac{f(y_n)}{f(w_n)}\right)^3\\
&+\frac{f(z_n)}{f(y_n)} + (4 - 2f[x_n, w_n])\frac{f(z_n)}{f(w_n)}.
\end{aligned}
\tag{4.3}
$$

Thukral eight order derivative free method(TEM):

$$
\begin{aligned}
y_n &= x_n - \frac{f(x_n)^2}{f(w_n) - f(x_n)},\\
z_n &= y_n - \frac{f[w_n, x_n] f(y_n)}{f[x_n, y_n] f[w_n, y_n]},\\
x_{n+1} &= z_n - \left(1 - \frac{f(z_n)}{f(w_n)}\right)^{-1} \left(1 + 2\frac{f(y_n)^3}{f(w_n)^2 f(x_n)}\right)\\
&\left(\frac{f(z_n)}{f[x_n, w_n] - f[x_n, y_n] + f[x_n, z_n]}\right),
\end{aligned}
\tag{4.4}
$$

where $w_n = x_n + \beta f(x_n)$, $n \in N$, $\beta \in R^+$. The test functions and their simple roots are listed in Table 1. All computations were performed in Mathematica 9.0 using variable precision arithmetic (VPA) to increase the number of significant digits. Here, we consider an approximate solution rather than the exact root, depending on the precision of the computer. Thus, we have the following stopping criterion $|f(x_n)| < 10^{-150}$. The computer specifications during numerical performance are Microsoft Windows 8 Intel Core i5-3210M CPU@ 2.50 GHz with 4.00 GB of RAM, 64-bit Operating System throughout this paper. In Table 3, we have observed that our contributed methods perform better in comparison with other existing seventh and eighth order methods and also obtain their COC using formula (1.4). The illustrative numerical results show that they agree with the theoretical results obtained in Theorems 1 and 2.

TABLE 1. Functions and their roots

Function	*Root*
$f_1(x) = sin3x + x\ cosx$	$\alpha_1 \approx 1.19776...$
$f_2(x) = (cosx^2)^{1/2} - logx.x^{1/2}$	$\alpha_2 \approx 1.21789...$
$f_3(x) = logx - x^{1/2} + 5$	$\alpha_3 \approx 8.30943...$
$f_4(x) = e^{sinx} - x + 1$	$\alpha_4 \approx 2.63066...$
$f_5(x) = e^{-x} - 1 + x/5$	$\alpha_5 \approx 4.96511...$
$f_6(x) = 2 - 3x + sinx^2$	$\alpha_6 \approx 0.91375...$

TABLE 2. Weight functions

Method 1	$A(t_n)$	$B(t_n)$	$H(u_n)$	
forms	$t^3 + \frac{1-t_n}{1-2t_n}$	$\frac{1-t_n}{1-3t_n}$	e^{u_n}	
Method 2	$A(t_n)$	$B(t_n)$	$H(u_n)$	$G(s_n)$
forms	$t_n^3 + \frac{1-t_n}{1-2t_n} - 8t_n^4$	$\frac{1-t_n}{1-3t_n} - 12t_n^3$	e^{u_n}	$\frac{1}{1-2s_n}$

Table 3. Comparison of numerical results for different derivative free methods

	n	TNFE	$\lvert f(x_n)\rvert$	COC
$f_1(x) = sin3x + xcosx, x_0 = 1$				
Method 1	3	12	10^{-257}	7
SKSM	3	12	10^{-20}	7
Method 2	3	12	10^{-496}	8
SKEM	-	-	-	div
TEM	3	12	10^{-288}	8
$f_2(x) = (cosx^2)^{1/2} - logx.x^{1/2}, x_0 = 1.2$				
Method 1	3	12	10^{-284}	7
SKSM	3	12	10^{-124}	7
Method 2	3	12	10^{-396}	8
SKEM	3	12	10^{-187}	8
TEM	3	12	10^{-300}	8
$f_3(x) = logx - x^{1/2} + 5, x_0 = 8$				
Method 1	3	12	10^{-234}	7
SKSM	3	12	10^{-54}	7
Method 2	3	12	10^{-309}	8
SKEM	3	12	10^{-49}	8
TEM	3	12	10^{-296}	8
$f_4(x) = e^{sinx} - x + 1, x_0 = 2.3$				
Method 1	3	12	10^{-344}	7
SKSM	3	12	10^{-104}	7
Method 2	3	12	10^{-525}	8
SKEM	3	12	10^{-138}	8
TEM	3	12	10^{-490}	8
$f_5(x) = e^{-x} - 1 + x/5, x_0 = 4.5$				
Method 1	3	12	10^{-539}	7
SKSM	3	12	10^{-515}	7
Method 2	3	12	10^{-745}	8
SKEM	3	12	10^{-736}	8
TEM	3	12	10^{-730}	8
$f_6(x) = 2 - 3x + sinx^2, x_0 = 0.6$				
Method 1	3	12	10^{-535}	7
SKSM	3	12	10^{-205}	7
Method 2	3	12	10^{-462}	8
SKEM	3	12	10^{-277}	8
TEM	3	12	10^{-438}	8

5 Conclusion

In this article, we have demonstrated the performance of a new seventh and eighth-order derivative-free methods. Convergence analysis proves that the new methods preserve their order of convergence and efficiency index. There are two major advantages of these three step seventh and eighth-order derivative free methods. First, we do not have to evaluate the derivatives of the functions. Therefore they are especially efficient where the computational cost of the derivative is expensive. Second the seventh and eighth order derivative free method gives better approximation of root compared to some other methods.

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